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# Uncapacitated Euclidean Hub Location: Strengthened Formulation, New Facets and a Relax-and-cut Algorithm

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**Abstract.** The multiple allocation uncapacitated hub location problem is considered. This problem arises in transportation systems when several locations send and receive passengers and/or express packages and the performance of these systems can be improved by using transshipment points (*hubs*), where the passengers/packages are collected and distributed.

An Integer Programming formulation, the one giving the best computational results until now, serves as a basis for the solution method. Using the fact that the transportation costs between hubs satisfy the triangle inequality, an analysis of the set of solutions that are not candidates to be optimal is carried out and, as a consequence, the formulation of the problem can be strengthened by means of powerful valid inequalities obtained through the study of the intersection graph of an associated set packing problem. The algorithm developed uses the most promising of these inequalities in a Lagrangian relaxation context to reduce the size of the branching tree and improve the computational times. This improvement is shown by means of a computational study, where a set of instances are optimally solved with low computational effort.

**Key words:** Facets, Hub location, Lagrangian relaxation, Set packing.

## 0. Introduction

Hub location problems arise in transportation systems when several locations send and receive passengers and/or express packages and the performance of these systems can be improved by using transshipment points (*hubs*), where the passengers/packages are collected and distributed.

Most of the papers in the literature devoted to the optimization of these systems are based on the use of Integer Programming techniques (IP). The papers cited in the following deal with two different versions of uncapacitated hub location; one of them corresponds with the problem studied in this paper, and the other one with the so-called p-hub median problem. But all of them are relevant in the development of new IP formulations for uncapacitated hub location. The first integer linear programming formulation for the uncapacitated multiple allocation hub location problem (the one dealt with in this paper) in the literature was given by Campbell (1994). Klincewicz (1996) used it to design a dual algorithm. Later, Skorin-Kapov et al. (1996) modified this formulation, using fewer constraints and the same number of variables, and O'Kelly et al. (1996) studied a second improvement which reduced still more the number of variables. Recently, Hamacher et al. (2001) and Cánovas et al. (2001) began to study the polyhedral structure of the problem. In Ernst and Krishnamoorthy (1998a, b), formulations based on a different approach were given.

In this paper, one of the known formulations of the problem will be improved, making use of the assumption that the transportation costs between hubs satisfy the triangle inequality, and a study of the structure of the associated polyhedron – the convex hull of the feasible solutions – will be carried out. In particular, several families of facets of this polyhedron will be obtained. Facets are non-dominated valid inequalities that can be used in a relax-and-cut environment to speed up the resolution of the problem, since when they are added to the linear relaxation of the problem (where the integrality constraints are removed), fractional solutions of this relaxation are cut off, and the performance of the relax-and-cut methods usually improves. However, there is another way of using the facets, by embedding them into a relaxation method, see for instance Guignard (1998). This is the approach used in this paper. It has several advantages, which will be detailed below. These facets remain valid inequalities when additional constraints are added to the formulation, and then the $p$-median version of the problem can also be improved.

The paper is organized as follows. In the next section, the details of the model are given. In Section 2, the known ILP formulations of UEHLP, which serve as a basis for the subsequent work, are reviewed. In Sections 3 and 4, the knowledge about the polyhedron associated with the set packing problem is applied to UEHLP. With the theoretical background of Sections 3 and 4, the solution method is specified in Section 5, and it is computationally tested in Section 6. Finally we outline future work and give some concluding remarks.

## 1. Details of the Model

Consider a set $N=\{1,\dots,n\}$ of locations – points, each of which receives a product (passengers, packages ... ) from all the other points and also sends the product to all the other points. Let $W_{ij} \geqslant 0$ be the amount of product to be sent from the $i$th point to the $j$th point for all $i,j \in N$, even when $i=j$. Also consider a set of locations $M=\{1,\dots,q\}$, which can be used as transshipment points (*hubs*) by paying a fixed cost denoted by $F_k \geqslant 0$, $k \in M$ (possibly different for each of them). Once the set of hubs is determined, the flow going from point $i$ to point $j$ must be sent through at least one hub.

Note that $W_{ii}$ can be greater than zero when the product must be sent to any hub to undergo any kind of process (manipulation, classification ...) and then returned to point $i$. If it is not the case, it suffices to fix $W_{ii}=0$ for all $i \in N$.

It is also assumed that: (i) associated with every two points $k, m \in M$, a cost $c_{km} \geqslant 0$ of transporting one unit of product from $k$ to $m$ is given; (ii) $c_{kk}=0$ $\forall k$; and (iii) associated with every two points $i \in N$, $k \in M$, two costs $b_{ik} \geqslant 0$ and $d_{ki} \geqslant 0$ of transport one unit of product from $i$ to $k$ and, respectively, from $k$ to $i$, are given (see Figure 1). It is assumed that the costs $c_{km}$ satisfy the triangle inequality. Hence, the product will be sent from $i \in N$ to $j \in N$ through one or, at most, two hubs. If the triangle inequality is not satisfied, additional constraints may be added to the model, if desired, to limit the number of hubs traversed by the product between a given origin and destination, see Cánovas et al. (2001), although this case is not considered here.

Finding the subset of points to be transformed into hubs and the hubs to be used in the route associated with each pair origin–destination in such a way that the total cost is minimized is what we call the *Uncapacitated Euclidean Hub Location Problem* (UEHLP, for brevity).

### 1.1. BASIC FORMULATION

Some formulations for different versions of uncapacitated hub location problems in the literature (the ones giving the best computational results until now) are based on four-indexed variables $x_{ijkm}$ representing the fraction of $W_{ij}$ which is routed through hubs $k$ and $m$ in this order (where perhaps $k=m$). What is called the *basic formulation* here is very similar to the formulation introduced by Campbell (1994). It reads

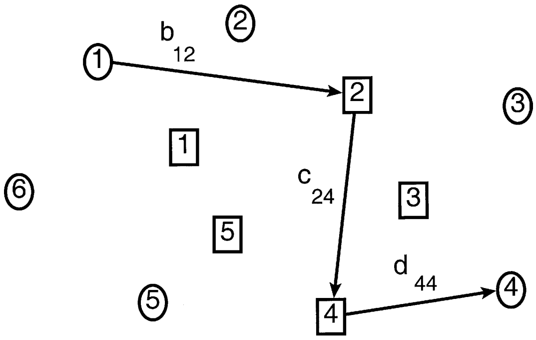


*Figure 1.* Transportation costs of the model. Points in the sets $N$ and $M$ are represented by circles and squares, respectively.

$$\text{(BUE)}\quad \min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{q}\sum_{m=1}^{q} W_{ij}C_{ijkm}x_{ijkm} + \sum_{k=1}^{q} F_k y_k \tag{1}$$

$$\text{s.t.}\ \sum_{k=1}^{q}\sum_{m=1}^{q} x_{ijkm} = 1 \quad \forall i, j \in N, \tag{2}$$

$$x_{ijkm} \leqslant y_k \quad \forall i, j \in N\ \forall k, m \in M, \tag{3}$$

$$x_{ijkm} \leqslant y_m \quad \forall i, j \in N\ \forall k, m \in M, \tag{4}$$

$$y_k \in \{0, 1\} \quad \forall k \in M,$$

$$x_{ijkm} \in \{0, 1\} \quad \forall i, j \in N\ \forall k, m \in M,$$

where the coefficients $C_{ijkm}$ are equal to $b_{ik} + c_{km} + d_{mj}$.

In this formulation, a binary variable $y_k$ takes the value 1 if and only if a hub is located at point $k$. Constraints (2) assure that all the flows are routed through a pair of hubs in $M$ (perhaps a pair $(k, k)$) and constraints (3) and (4) guarantee that the cost associated to a hub is paid if it is used. We recall that it suffices to consider routes in which the passengers traverse at most two hubs since the triangle inequality, satisfied by the costs $c_{km}$, guarantees that there will be no more than two hubs in any route of the optimal solution.

(BUE) is a large formulation with a very weak linear relaxation. Thus, solving UEHLP by means of branching methods based on the lower bounds obtained from the linear relaxation of (BUE) is a difficult task. The same difficulties arise when using dual methods based on similar formulations, see e.g. Klincewicz (1996).

In order to improve this formulation, some ideas from the field of the polyhedral structure of set packing problems can be used. It is necessary now to introduce some background on this problem.

### 1.2. REFORMULATING UEHLP AS A SET PACKING PROBLEM

A *set packing problem* is a binary optimization problem

$$\text{(SPP):}\quad \text{Opt } \{ct :\ \ At \leqslant \mathbf{1}_r,\ \ t \in \{0, 1\}^s\},$$

where $c \in \mathbf{R}^s$, $A \in \{0, 1\}^{r\times s}$ and $\mathbf{1}_r$ is an $r$-vector of ones.

In order to formulate UEHLP as a set packing problem, a reasoning presented in Cho *et al.* (1983) for the simple plant location problem is adapted. Take binary variables $y'_k = 1 - y_k\ \forall k$, take a huge number $M\sum_i\sum_j W_{ij}$, rewrite it using (2) as $\sum_i\sum_j\sum_k\sum_m MW_{ij}x_{ijkm}$ and subtract it from the objective function (1), obtaining

$$\sum_k F_k - \sum_k F_k y'_k - \sum_i \sum_j \sum_k \sum_m W_{ij}(M - C_{ijkm})x_{ijkm}.$$

Now, the objective function can be replaced by the maximization of a linear function with positive coefficients after removing the irrelevant constant $\sum_k F_k$:

$$\text{(SPUE)} \quad \max \quad \sum_k F_k y'_k + \sum_i \sum_j \sum_k \sum_m W_{ij} C'_{ijkm} x_{ijkm} \tag{5}$$

$$\text{s.t.} \quad \sum_k \sum_m x_{ijkm} \leqslant 1 \quad \forall i, j, \tag{6}$$

$$x_{ijkm} + y'_k \leqslant 1 \quad \forall i, j, k, m, \tag{7}$$

$$x_{ijkm} + y'_m \leqslant 1 \quad \forall i, j, k, m, \tag{8}$$

$$y'_j \in \{0, 1\} \quad \forall j, \tag{9}$$

$$x_{ijkm} \in \{0, 1\} \quad \forall i, j, k, m, \tag{10}$$

where $C'_{ijkm} = M - C_{ijkm} > 0 \ \forall i, j, k, m$. Note that equalities (2) have been replaced by inequalities (6). This can be done due to the following result.

PROPOSITION 11. Any optimal solution of (SPUE) satisfies all constraints (6) as equalities.

*Proof.* Consider any feasible solution not satisfying all constraints (6) as equalities. Then, for some $i_1$ and some $j_1$, $\sum_k \sum_m x_{i_1 j_1 km} = 0$ holds. Modify this solution, taking $x_{i_1 j_1 11} = 1$, $y'_1 = 0$ to obtain a new feasible solution. In the worst case, if $y'_1$ took the value one in the first optimal solution, the increase of the value of the objective function is $C'_{i_1 j_1 11} - F_1 = M - C_{i_1 j_1 11} - F_1$. Since $M$ is huge, this is greater than zero, and the first solution could not be optimal. □

## 2. Strengthening (SPUE)

The following two theorems are the basis of the forthcoming results. They take into account the structure of any optimal solution of UEHLP in order to eliminate part of the feasible solutions which are not candidates for the optimum. In Figure 2, the structure of the optimal solution with respect to the flow with origin in point 1 is shown. As demonstrated in the two following theorems, the arcs traversed by this flow are tree-shaped. Similarly, the arcs traversed by flows with destination in a given point $j$ are tree-shaped.

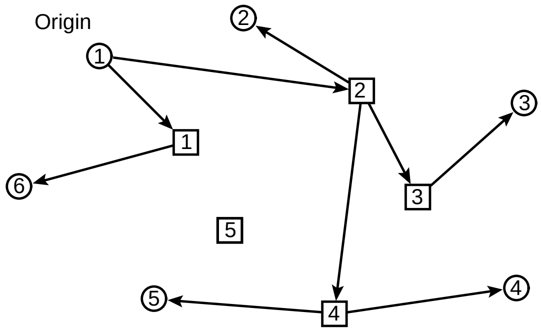


*Figure 2.* The structure of the routes of the product coming from point 1, in an optimal solution of UEHLP. Points in the sets $N$ and $M$ are represented by circles and squares, respectively.

THEOREM 1. *(SPUE) always has an optimal solution satisfying*

$$x_{ijkm} = 1 \Rightarrow x_{ij'k'm} = 0 \tag{11}$$

*for all* $i$, $j$, $j'$, $m$, *and all* $k \neq k'$ *and*

$$x_{ijkm} = 1 \Rightarrow x_{i'jkm'} = 0 \tag{12}$$

*for all* $i$, $i'$, $j$, $k$, *and all* $m \neq m'$.

*Proof.* Take any optimal solution of (SPUE), and assume $x_{ijkm} = 1 = x_{ij'k'm}$ for certain $i$, $j$, $j'$, $m$ and $k \neq k'$. Then, in this optimal solution

$$y'_k = y'_{k'} = y'_m = 0$$

holds. One of the costs associated with the routes $i \to k \to m$ and $i \to k' \to m$ must be greater than or equal to the other one, say $b_{ik} + c_{km} \leqslant b_{ik'} + c_{k'm}$. Therefore, the unit coefficient associated with variable $x_{ij'km}$, $C'_{ij'km}$, is greater than or equal to the unit cost associated with the variable $x_{ij'k'm}$, $C'_{ij'k'm}$. Since $y'_k = y'_m = 0$, by replacing $x_{ij'k'm}$ by 0 and $x_{ij'km}$ by 1 in the optimal solution, another feasible solution with no lower cost (i.e., also optimal) is obtained. Proceeding iteratively, the last optimal solution which is obtained satisfies (11). By symmetry, (12) also holds. □

Therefore, the optimal solution never includes (i) two paths from the same origin through different first hubs to the same second hub, nor (ii) two paths from the same first hub through different second hubs to the same destination. Now the following constraints can be added to the formulation:

$$x_{ijkm} + x_{ij'k'm} \leqslant 1 \quad \forall i, j, j', m, k \neq k', \tag{13}$$

$$x_{ijkm} + x_{i'jkm'} \leqslant 1 \quad \forall i, i', j, k, m \neq m'. \tag{14}$$

THEOREM 2. *(SPUE) always has an optimal solution satisfying*

$$x_{ijkm} = 1 \Rightarrow x_{ij'\ell k} = 0 \tag{15}$$

*for all* $i$, $j$, $j'$, $m$, *and all* $k \neq \ell$ *and*

$$x_{ijkm} = 1 \Rightarrow x_{i'jm\ell} = 0 \tag{16}$$

*for all* $i$, $i'$, $j$, $k$, *and all* $m \neq \ell$.

*Proof.* Take any optimal solution of (SPUE), and assume $x_{ijkm} = 1 = x_{ij'\ell k}$ for certain $i$, $j$, $j'$, $m$ and $k \neq \ell$. Then, in this optimal solution

$$y'_k = y'_\ell = y'_m = x_{ij'kk} = 0$$

holds. Since in the optimal solution $x_{ij'kk} = 0$ and $x_{ij'\ell k} = 1$ hold, the cost $b_{i\ell} + c_{\ell k}$ must be less than or equal to the cost $b_{ik}$. Then, $b_{i\ell} + c_{\ell k} + c_{km} + d_{mj} \leqslant b_{ik} + c_{km} + d_{mj}$ holds. Using the triangle inequality, it follows that $b_{i\ell} + c_{\ell m} + d_{mj} \leqslant b_{ik} + c_{km} + d_{mj}$. Therefore, the unit coefficient $C'_{ij\ell m}$ is greater than or equal to the unit coefficient $C'_{ijkm}$. Since $y'_\ell = y'_m = 0$, by replacing $x_{ijkm}$ by 0 and $x_{ij\ell m}$ by 1 in the optimal solution, another feasible solution with no lower cost (i.e., also optimal) is obtained. Proceeding iteratively, the last optimal solution which is obtained satisfies (15). By symmetry, (16) holds. □

Hence, (i) if in the optimal solution there exists one path going directly from a given origin to a given hub, there is no path from this origin to this hub through a second hub; and (ii) if in the optimal solution there exists one path going directly from a given hub to a given destination, there is no path from this hub to this destination through a second hub. Consequently, the following constraints can be added to the formulation:

$$x_{ijkm} + x_{ij'\ell k} \leqslant 1 \quad \forall i, j, j', m, k \neq \ell, \tag{17}$$

$$x_{ijkm} + x_{i'jm\ell} \leqslant 1 \quad \forall i, i', j, k, m \neq \ell. \tag{18}$$

Therefore, the following strengthened formulation, termed (UEHLP), is still a set packing problem formulation valid for UEHLP:

$$\text{(UEHLP)} \ \max\{(5) \text{ s.t. } (13), (14), (17), (18), (6), (7), (8), (9), (10)\}.$$

## 3. The Intersection Graph

Before continuing, it is necessary to introduce some background on graphs associated with SPP.

Let $G=(V,E)$ be a graph with node set $V$ and edge set $E$. A nonempty subset of $V$ of mutually non-adjacent nodes in $G$ is called a *packing*. The *neighborhood* $N(v)$ of a node $v$ is the set of nodes that are adjacent to $v$. The *incidence vector* of a subset $B$ of $V$ is a binary vector $(t_1,\dots,t_{|V|})$ where $t_j=1$ if and only if the $j$th node of $V$ belongs to $B$, $j=1,\dots,|V|$. $P_I(G)$ is the set of incidence vectors of all the packings of $G$, and the *polytope associated with* $G$, $P(G)$, is the convex hull of $P_I(G)$.

The *graph associated with* (*intersection graph of*) (SPP) is $G=(V,E)$ with $|V|=s$ and $(v_i,v_j)\in E$ if and only if the $i$th and $j$th columns of $A$ are not orthogonal. Then, if $G$ is the graph associated with (SPP), the feasible set of (SPP) is $P_I(G)$ and the optimal solutions of (SPP) can be obtained by solving the linear optimization problem

$$\text{Opt}\ \{ct:\quad t\in P(G)\}.$$

Call the intersection graph of (UEHLP) $G(n,q)$, and note $P^{nq}:=P(G(n,q))$.

Figure 3 shows, in graph $G(3,4)$, the nodes in the neighborhood of $x_{1223}$ (left hand side) and the nodes in the neighborhood of $x_{1233}$ (right hand side). In the figures of this paper, nodes associated with $x$-variables ($x$-nodes) are represented by rectangles which are arranged in $n\times n$ blocks of $q\times q$ rectangles. The block in the first row of blocks and second column of blocks is associated with variables $x_{12km}$ for all $k$ and all $m$ in $M$, which are arranged in a matrix-like structure. The $y$-node associated with variable $y'_k$ will be represented by the $k$th circle. It is assumed throughout the paper that $n,q\geqslant 3$. Accordingly, the black-filled node in the left hand side of Figure 3 is associated with $x_{1223}$, while the nodes in the neighborhood of $x_{1223}$ appear in bold face.

Taking into account the structure of $G(n,q)$, i.e., that changing the order of the points of $N$ and/or $M$ an isomorphic graph is obtained, then any subgraph of $G(n,q)$ can be represented using the first boxes and the first rows and columns inside these boxes.

## 4. Clique Facets

Associated with the new formulation (UEHLP), arises the convex hull of its feasible solutions (the polytope $P^{nq}$). Knowing a set of hyperplanes which characterize this polytope enable the problem to be solved by simply maximizing the objective function on these linear inequalities (without integrity constraints). In fact, only a few of these hyperplanes can be

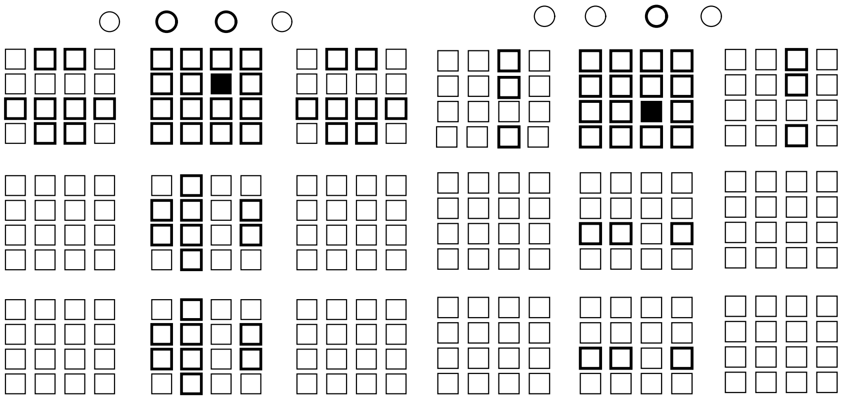

*Figure 3.* Representation of the intersection graph of (UEHLP) when $n=3$, $q=4$.

obtained but, by combining branching techniques with the partial knowledge of the polytope, medium sized instances can be solved.

Non-dominated inequalities are preferred to avoid redundant useless information about the polytope. Some concepts that clarify this aspect of the study are now introduced.

A linear inequality $\rho t \leqslant \rho_0$ is said to be *valid* for $P(G)$ if it holds for all $t \in P(G)$. A valid inequality for $P(G)$ is a *facet* of $P(G)$ if and only if it is satisfied as an equality by $|V|$ independent vertices of $P(G)$. A *complete graph* is that in which all the nodes are pairwise adjacent. A *clique* in $G$ is a maximal complete subgraph.

The simplest facets of (UEHLP) are those inequalities having binary coefficients and right hand side 1. The following well-known result will be used.

THEOREM 1 (Nemhauser and Trotter (1974), Padberg (1973, 1977)). *Let $G=(V,E)$ be a graph and let $B$ be a subset of $V$. The inequality $\sum_{j\in B} t_j \leqslant 1$ is a facet-defining inequality of $P(G)$ if and only if the subgraph induced by $B$ is a clique in $G$.*

The task to be carried out is to identify the cliques in $G(n,q)$, and then the corresponding *clique facets* will directly be identified.

Since $y'$-nodes are not interconnected, a clique in the intersection graph $G(n,q)$ contains either one or zero $y'$-nodes.

THEOREM 2. *The inequalities*

$$y'_a + \sum_{k=1}^{q} x_{ij(a)ak} + \sum_{k\neq a} x_{ij(k)ka} \leqslant 1 \tag{19}$$

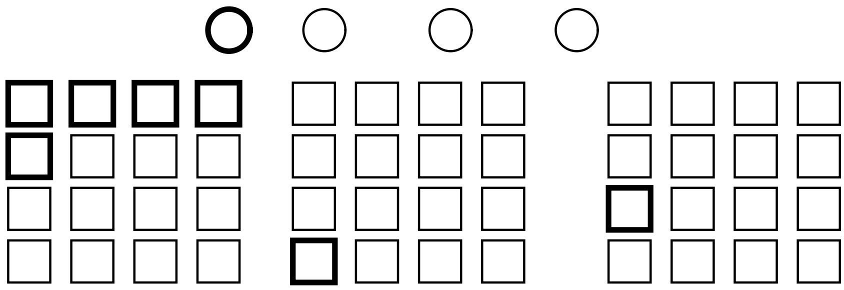

*Figure 4.* Cliques in the intersection graph containing one $y$-node.

*with $a \in M$, $i \in N$, $j(k)$ an index in $N$ associated with each $k \in M$ and*

$$y'_a + \sum_{k=1}^{q} x_{i(a)jka} + \sum_{k \neq a} x_{i(k)jak} \leqslant 1 \tag{20}$$

*with $a \in M$, $j \in N$, $i(k)$ an index in $N$ associated with each $k \in M$, are the unique clique facets of $P^{nq}$ containing one $y'$-node.*

*Proof.* Cliques containing one $y'$-node cannot contain more $y'$-nodes. If one $x$-node belongs to the clique, other $x$-nodes with the same first index (resp. second index), lying in the same column and different rows (resp. the same row and different columns) may belong to the clique. Finally, a complete row (resp. column) in one of the boxes associated with the first index (resp. second index) must be added to obtain a maximal complete subgraph of $G(n, q)$, in the shape of (19) (resp. (20)). □

The cliques (19) obtained in Theorem 2 are illustrated in Figure 4.

All the clique facets which are going to be obtained in the rest of the paper are pairwise symmetric. Although we list all of them in the theorems, only one of the components of each symmetric pair (the one associated with a common first index $i$) is going to be demonstrated.

The formulation of UEHLP in Cánovas et al. (2001) and Hamacher et al. (2001) includes the particular case of facets (19) in which $j(k) = j\ \forall k$, as well as the particular case of facets (20) in which $i(k) = i\ \forall k$. The rest of facets listed in Theorem 2, however, are derived from the new constraints (13), (14), (17) and (18).

THEOREM 3. *The inequalities*

$$\sum_{k=1}^{q} \sum_{m=1}^{q} x_{ijkm} \leqslant 1 \tag{21}$$

*with $i, j \in N$, are the unique clique facets of $P^{nq}$ not containing $y'$-nodes and containing all the $x$-nodes in the same box.*

*Proof.* Since all the $x$-nodes in the same box are interconnected, the result is obvious. □

THEOREM 4. *The inequalities*

$$\sum_{m=1}^{q} x_{ij_1am} + \sum_{k\neq a,b} x_{ij_1ka} + \sum_{k\neq a,b} x_{ij_1kb} + x_{ij_2ba} \leqslant 1 \tag{22}$$

*with $i, j_1, j_2 \in N$, $j_1 \neq j_2$, $a, b \in M$, $a \neq b$, and*

$$\sum_{k=1}^{q} x_{i_1jka} + \sum_{m\neq a,b} x_{i_1jam} + \sum_{m\neq a,b} x_{i_1jbm} + x_{i_2jab} \leqslant 1 \tag{23}$$

*with $j, i_1, i_2 \in N$, $i_1 \neq i_2$, $a, b \in M$, $a \neq b$, are the unique clique facets of $P^{nq}$ not containing $y'$-nodes and containing $x$-nodes in the diagonal of some box as well as $x$-nodes in some other box(es).*

*Proof.* Consider a first node $x_{ij_1aa}$, and a second node in a different box $j_2$ associated with the same origin $i$ and linked to $x_{ij_1aa}$: $x_{ij_2ba}$, $b \neq a$. Since $y_a$ does not belong to the facet, another $x$-node in box $j_1$ must be included in it, and, then, on adding the nodes connected to these three $x$-nodes, a facet in the shape of (22) arises. □

The cliques (23) obtained in Theorem 4 are illustrated in Figure 5.

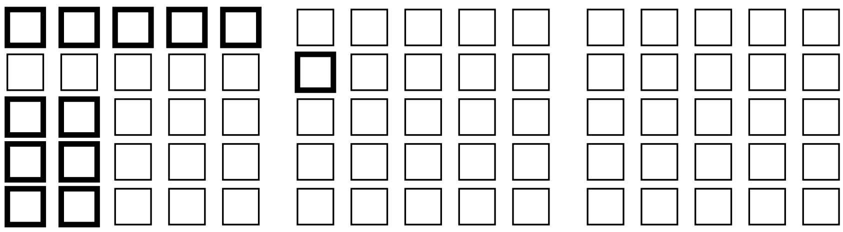

*Figure 5.* Cliques in the intersection graph not containing $y'$-nodes, containing $x$-nodes in the diagonal of some box and containing $x$-nodes in more than one box

THEOREM 5. The inequalities

$$\sum_{k\neq a,b} x_{ij_1ka} + \sum_{k\neq a,b} x_{ij_1kb} + x_{ij_2ab} + x_{ij_3ba} \leqslant 1, \tag{24}$$

$$\sum_{k\neq a,c} x_{ij_1ka} + x_{ij_1bc} + x_{ij_2ab} + x_{ij_2ac} + x_{ij_3ca} \leqslant 1, \tag{25}$$

$$x_{ij_1ab} + x_{ij_1ac} + x_{ij_2ba} + x_{ij_2bc} + x_{ij_3ca} + x_{ij_3cb} \leqslant 1 \tag{26}$$

with $a,b,c\in M,\ a\neq b\neq c\neq a,\ i, j_1, j_2, j_3\in N,\ j_1\neq j_2\neq j_3\neq j_1$, and

$$\sum_{m\neq a,b} x_{i_1jam} + \sum_{m\neq a,b} x_{i_1jbm} + x_{i_2jba} + x_{i_3jab} \leqslant 1, \tag{27}$$

$$\sum_{m\neq a,c} x_{i_1jam} + x_{i_1jcb} + x_{i_2jba} + x_{i_2jca} + x_{i_3jac} \leqslant 1, \tag{28}$$

$$x_{i_1jba} + x_{i_1jca} + x_{i_2jab} + x_{i_2jcb} + x_{i_3jac} + x_{i_3jbc} \leqslant 1 \tag{29}$$

with $a,b,c\in M,\ a\neq b\neq c\neq a,\ j,i_1,i_2,i_3\in N,\ i_1\neq i_2\neq i_3\neq i_1$, are the unique clique facets of $P^{nq}$ not containing $y'$-nodes nor $x$-nodes in the diagonal of any box, and containing $x$-nodes in at least three different boxes.

*Proof.* Consider a first node $x_{ij_1ca}$, $c\neq a$, and a second node in a different box associated with the same origin $i$. There are three possibilities:

1. The second node is in column $a$, i.e. it is $x_{ij_2ba}$ with $b\neq a, c$. Then, since all the nodes in a third box, as well as $x_{ij_1ca}$ and $x_{ij_2ba}$, are connected to $y'_a$, a node not connected to $y'_a$, lying in one of the two used boxes, must be added to the facet. There are two non-symmetric possibilities:
   (a) Take a node in the first box in column $b$, $x_{ij_1db}$ with $d\neq a,b,c$. Hence, the node to be added in the third box must be one of the two following choices:
      (i) $x_{ij_3ab}$; then, if $x_{ij_2bc}$ or $x_{ij_3ad}$ form part of the graph, a facet in the shape of (24) is obtained; otherwise, a facet in the shape of (25) is obtained.
      (ii) $x_{ij_3ad}$; then, a facet in the shape of (25) is directly obtained.
   (b) Take the node in the first box $x_{ij_1cb}$. Hence, the node to be added in the third box must be one of the two following choices:
      (i) $x_{ij_3ab}$; then, if no more nodes from out of box $j_1$ are added, facet (24) is obtained; if nodes in only one of the boxes $j_2$ and $j_3$ are added, facet (25) is obtained; and, finally, if nodes are added in both boxes $j_2$ and $j_3$, a facet in the shape of (26) is obtained.

(ii) $x_{ij_3ac}$ is taken and $x_{ij_3ab}$ is not. Then, a facet in the shape of (25) is directly obtained.

2. The second node is in column $c$, $x_{ij_2bc}$ with $b \neq a, c$, and no nodes in column $a$ are added to the facet. There are three possibilities:
   (a) The node in the third box is $x_{ij_3ab}$, then if $x_{ij_3ac}$ is added, a facet in the shape of (25) is obtained and, if $x_{ij_3ac}$ is not added, both (24) and (25) are possible.
   (b) The node in the third box is $x_{ij_3ac}$ is added, and $x_{ij_3ab}$ is not added. Hence, all the remaining nodes are connected to $y'_c$.
   (c) The node in the third box is $x_{ij_3dc}$ with $d \neq a, b, c$ and nodes $x_{ij_3ac}$ and $x_{ij_3ac}$ are not added. The only (except symmetries) way to avoid the connection with $y'_c$ is by adding $x_{ij_2ad}$, then the remaining nodes form a complete subgraph and this directly leads to facet (25).
3. The second node is in row $a$, $x_{ij_2ab}$, $b \neq a, c$, and there are no nodes in columns $a$ and $c$. Hence, the only remaining node in a third box is $x_{ij_3bc}$, and only facets in the shape of (24) or (25) are possible.

If the second node is taken from inside a box associated with the same destination, the symmetric facets (27–29) are obtained. □

The cliques (24)–(26) obtained in Theorem 5 are illustrated in Figure 6.

THEOREM 6. The inequalities

$$\sum_{k \neq a} x_{ij_1ka} + x_{ij_1ba} + x_{ij_1ca} + x_{ij_2ab} + x_{ij_2ac} \leqslant 1, \tag{30}$$

$$\sum_{k \neq a,b} x_{ij_1ka} + x_{ij_1cb} + x_{ij_2ab} + x_{ij_2ac} + x_{ij_2ba} \leqslant 1, \tag{31}$$

$$x_{ij_1ac} + x_{ij_1bd} + x_{ij_2dc} + x_{ij_2da} + x_{ij_2cd} + x_{ij_2cb} \leqslant 1, \tag{32}$$

$$\sum_{k \neq a,b} x_{ij_1ka} + \sum_{k \neq a,b} x_{ij_1kb} + x_{ij_2ab} + x_{ij_2ba} \leqslant 1 \tag{33}$$

with $i, j_1, j_2 \in N$, $j_1 \neq j_2$, $a, b, c, d \in M$ different, and

$$\sum_{m \neq a} x_{i_1jam} + x_{i_1jab} + x_{i_1jac} + x_{i_2jba} + x_{i_2jca} \leqslant 1, \tag{34}$$

$$\sum_{m \neq a,b} x_{i_1jam} + x_{i_1jbc} + x_{i_2jba} + x_{i_2jca} + x_{i_2jab} \leqslant 1, \tag{35}$$

$$x_{i_1jca} + x_{i_1jdb} + x_{i_2jcd} + x_{i_2jad} + x_{i_2jdc} + x_{i_2jbc} \leqslant 1, \tag{36}$$

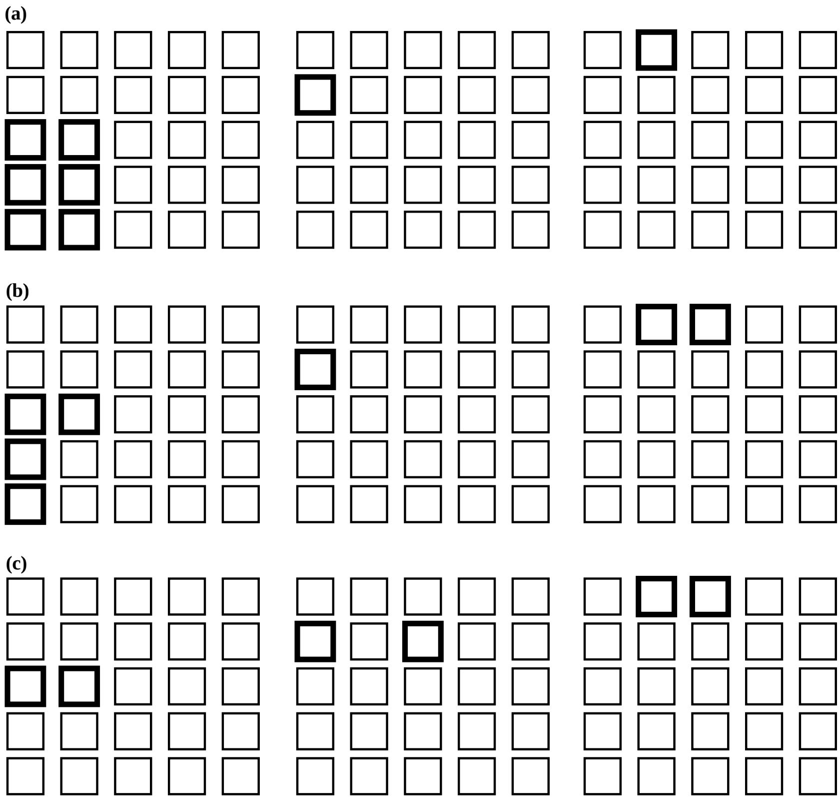

*Figure 6.* Cliques in the intersection graph not containing $y'$-nodes nor $x$-nodes in the diagonal of any box, and containing $x$-nodes in at least three different boxes.

$$\sum_{m \neq a,b} x_{i_1 jam} + \sum_{m \neq a,b} x_{i_1 jbm} + x_{i_2 jab} + x_{i_2 jba} \leqslant 1 \tag{37}$$

with $j, i_1, i_2 \in N$, $i_1 \neq i_2$, $a, b, c, d \in M$ different, are the only clique facets of $P^{nq}$ not containing $y'$-nodes nor $x$-nodes in the diagonal of any box, and containing $x$-nodes in exactly two different boxes.

*Proof.* Consider a first node $x_{ij_1ca}$, $c \neq a$, and a second node in a different box associated with the same origin $i$. There are four possibilities:

1. The second node is $x_{ij_2ac}$. Then, since $x_{ij_1ca}$ and $x_{ij_1ac}$ are connected to $y'_a$ and $y'_c$, nodes not connected to $y'_a$ and $y'_c$ must be added to the facet. There are three non-symmetric possibilities:
   (a) Take a node in the first box in column $a$, $x_{ij_1ba}$ with $b \neq a, c$. Then, to avoid the connection with $y'_a$ another node in box $j_1$ must be added. There are three possibilities:

(i) $x_{ij_1dc}$; then, only $x_{ij_2ad}$ remains in the second box, and a facet in the shape of (30) is obtained.
(ii) $x_{ij_1bc}$ or $x_{ij_1cb}$; then, in the second box only $x_{ij_2ab}$ remains, and a facet in the shape of (30) is again obtained.
(iii) $x_{ij_1cd}$; then, if $x_{ij_2ad}$ is added, a facet in the shape of (30) arises and, in other cases, a facet in the shape of (31) arises.

(b) Take the node in the second box $x_{ij_2ba}$. Then, the only node remaining in the first box is $x_{ij_1cb}$ and (30) again arises.
(c) Take the node in the second box $x_{ij_2ab}$. There are two cases:

(i) Add $x_{ij_1cb}$; then, facet (31) is obtained.
(ii) Add $x_{ij_1bc}$; no maximal complete subgraph can be obtained.

2. The second node is in row $a$, $x_{ij_2ab}$ with $b \neq a, c$, and $x_{ij_2ac}$ is not added to the facet. There are four ways to avoid the connection with $y'_a$ without limiting the number of nodes in the first box to one:

(a) Add $x_{ij_1bc}$, then $x_{ij_1cb}$ is connected to all the used and remaining nodes and must be added to the facet, a second node in the second box $x_{ij_2dc}$ with $d \neq a, b, c$ must be added, and the only possibility to obtain a maximal complete subgraph is to add $x_{ij_1bd}$, leading to facet (32).
(b) Add $x_{ij_1bd}$, with $d \neq a, b, c$. Then, if $x_{ij_2ad}$ is added to the subgraph, a facet in the shape of (31) or (32) is obtained. In any other case, no maximal complete subgraph can be obtained.
(c) Add $x_{ij_1ab}$, and do not add any node in the row $b$ of the first box. There are three possibilities:

(i) Add $x_{ij_2dc}$, then a facet in the shape of (30) is obtained.
(ii) Add $x_{ij_2bc}$, then a facet in the shape of (31) is obtained.
(iii) Add $x_{ij_2ba}$, then a facet in the shape of (33) is obtained.

(d) Add $x_{ij_1db}$ for some $d \neq a, b, c$, and do not add any node in the row $b$ of the first box nor $x_{ij_1cb}$. Then, $x_{ij_2ba}$ must be added and, to obtain a maximal complete subgraph, either $x_{ij_2bc}$ or $x_{ij_2ad}$ must be added. In both cases, a facet in the shape of (32) is obtained.

3. The second node is in column $a$ of $j_2$, and no nodes in the row $a$ of $j_2$ are added. To avoid the connection to $y'_a$ there are only two non-symmetric cases:

(a) Add $x_{ij_1cb}$. Then, if $x_{ij_2bc}$ is added, facets in the shape of (30) and (31) can be obtained and if $x_{ij2dc}$ for some $d \neq a, b, c$ is added, a facet in the shape of (32) is obtained.
(b) Add $x_{ij_1db}$ for some $d \neq a, b, c$, but then no maximal complete subgraph can be obtained.

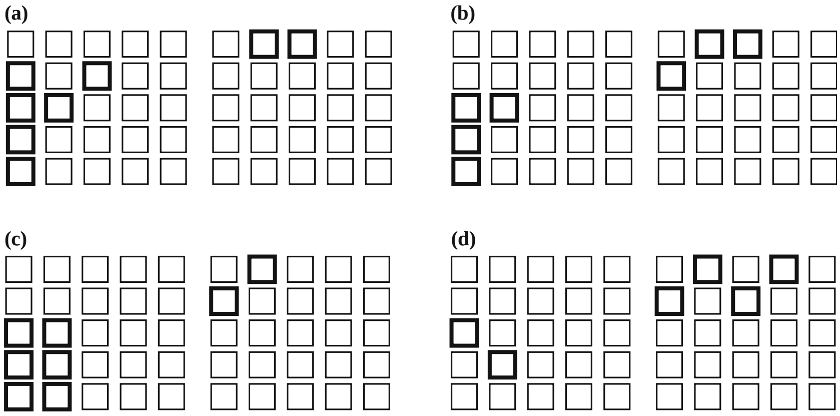


*Figure 7.* Cliques in the intersection graph containing neither $y'$-nodes nor $x$-nodes in the diagonal of any box, and containing $x$-nodes in exactly two different boxes.

4. The second node is in column $c$ of box $j_2$, and the row and column $a$ of $j_2$ are not used. Then, all the used and remaining nodes are connected to $y'_c$.

If the second node is taken from inside a box associated with the same destination, the symmetric facets (34–37) are obtained. □

The cliques (30–33) obtained in Theorem 6 are illustrated in Figure 7.

## 5. Solution approach

UEHLP will be solved by a relax-and-cut algorithm, specifically designed to make use of the families of facets obtained in the previous section. The objective function (1) of (BUE) is going to be used – then, the minimization version of the problem is considered again, and constraints (2) will be kept in the subproblem, while constraints (3) and (4) will not be used anymore. To improve the lower bounds needed to fathom nodes, new facets will be progressively incorporated to the subproblem. These facets are chosen from among those violated by the optimal solution associated with the Lagrangian multipliers used in the last iteration of a subgradient procedure. It is common in the literature to use the linear relaxation and, in that case, the separation problem is to find a facet violated by a fractional solution of the continuous problem. Instead, it is the Lagrangian relaxation which is going to be used here, and hence the separation problem is to find a facet violated by the integer optimal solution of the relaxed subproblem, which is infeasible with respect to the primal problem. Since this integer solution will satisfy constraints (2), it will be easy to keep track of it by simply saving the collection $(k_{ij}, m_{ij})$ of pairs of hubs associated with the pairs origin-destination $(i, j)$; this is going to be a great advantage when solving the separation problem.

### 5.1. THE SUBPROBLEM IN A NODE OF THE BRANCHING TREE

Some modifications of the basic formulation are needed to represent a node of the branching tree, since some hubs will have been opened and some other hubs will have been closed. To be precise, let $M_1 = \{k: \ y_k = 1\}$ be the set of opened hubs, let $M_0 = \{k: \ y_k = 0\}$ be the set of closed hubs, and let $M_2 = M - M_0 - M_1$ be the set of free hubs. Some additional inequalities, obtained using Theorems 2 and 4–6, will also have been added to the formulation.

Moreover, some $x$-variables are going to be fixed to zero at the beginning of the process. Define, for each $i, j, k, m$, $\hat{C}_{ijkm} = \min\{C_{ijkm}, \ C_{ijmk}, \ C_{ijkk}, \ C_{ijmm}\}$ and

$$\theta = \{x_{ijkm} \quad \text{s.t.} \quad C_{ijkm} > \hat{C}_{ijkm}\}.$$

$x$-variables in $\theta$ always take value 0 at any optimal solution of (UEHLP), and they can be removed from the formulation (see Cánovas et al., 2001). Consequently, the considered subproblem is in the form of

$$\begin{aligned}
&\text{(P)} \ \sum_{a \in M_1} F_a + && (38)\\
&\min \sum_i \sum_j \sum_k \sum_m W_{ij} C_{ijkm} x_{ijkm} + \sum_{a \in M_2} F_a y_a \\
&\text{s.t.} \ \sum_k \sum_m x_{ijkm} = 1 \quad \forall i, j \in N, \\
&\sum_i \sum_j \sum_k \sum_m \pi^{ta}_{ijkm} x_{ijkm} \leqslant y_a \quad \forall a \in M_2 \ \forall t \in T^a_y, && (39)\\
&\sum_i \sum_j \sum_k \sum_m \mu^{t}_{ijkm} x_{ijkm} \leqslant 1 \quad \forall t \in T_x, && (40)\\
&y_k \in \{0, 1\} \quad \forall k \in M_2, \\
&x_{ijkm} = 0 \quad \forall i, j \in N \ \forall k, m \in M_0, \\
&x_{ijkm} = 0 \quad \forall (i, j, k, m) \in \theta, \\
&x_{ijkm} \in \{0, 1\} \quad \forall (i, j, k, m) \in \Theta,
\end{aligned}$$

where $\Theta = \{(i, j, k, m): \ i, j \in N, \ k, m \in M_1 \cup M_2, \ (i, j, k, m) \notin \theta\}$.

In the node considered, it is assumed that constraints (39) have been selected from among the facets in families (19) and (20), and constraints (40) have been selected from among the other families of facets obtained in Section 4. Note that it is possible to fathom a node of the branching tree by adding constraints of type (39) containing only some of the $y$-variables.

Constraints (39) and (40) are going to be relaxed through Lagrange multipliers $u^t_a \geqslant 0$, $a \in M_2$, $t \in T^a_y$ and $v^t \geqslant 0$, $t \in T_x$. The Lagrangian relaxed problem (PRuv) is then obtained, with objective function

$$\begin{aligned}
&\sum_{a\in M_1} F_a+\sum_i\sum_j\sum_k\sum_m W_{ij}C_{ijkm}x_{ijkm}+\sum_{a\in M_2} F_a y_a\\
&\quad+\sum_{a\in M_2}\sum_{t\in T_y^a} u_a^t\left(\sum_i\sum_j\sum_k\sum_m \pi_{ijkm}^{ta}x_{ijkm}-y_a\right)\\
&\quad+\sum_{t\in T_x} v^t\left(\sum_i\sum_j\sum_k\sum_m \mu_{ijkm}^{t}x_{ijkm}-1\right)=\sum_{a\in M_1} F_a-\sum_{t\in T_x} v^t\\
&\quad+\sum_i\sum_j\sum_k\sum_m\left(W_{ij}C_{ijkm}+\sum_{a\in M_2}\sum_{t\in T_y^a} u_a^t\pi_{ijkm}^{ta}+\sum_{t\in T_x} v^t\mu_{ijkm}^t\right)x_{ijkm}\\
&\quad+\sum_{a\in M_2}\left(F_a-\sum_{t\in T_y^a} u_a^t\pi_{ijkm}^{ta}\right)y_a.
\end{aligned}$$

(PRuv) decomposes in a natural way in the constant $\sum_{a\in M_1} F_a-\sum_{t\in T_x} v^t$ plus two subproblems in $x$ and $y$ in the form of

$$\begin{aligned}
\text{(PRuvx)}\ \min\ &\sum_i\sum_j\sum_k\sum_m G_{ijkm}x_{ijkm}\\
\text{s.t.}\ &\sum_k\sum_m x_{ijkm}=1 \quad \forall i,j\in N,\\
&x_{ijkm}=0 \quad \forall i,j\in N\ \forall k,m\in M_0,\\
&x_{ijkm}=0 \quad \forall (i,j,k,m)\in\theta,\\
&x_{ijkm}\in\{0,1\} \quad \forall (i,j,k,m)\in\Theta,
\end{aligned}$$

where $G_{ijkm}=W_{ij}C_{ijkm}+\sum_{a\in M_2}\sum_{t\in T_y^a} u_a^t\pi_{ijkm}^{ta}+\sum_{t\in T_x} v^t\mu_{ijkm}^t$, $i,j\in N$, $k,m\in M$, and

$$\begin{aligned}
\text{(PRuvy)}\ \min\ &\sum_{a\in M_2}\left(F_a-\sum_{t\in T_y^a} u_a^t\pi_{ijkm}^{ta}\right)y_a\\
\text{s.t.}\ &y_k\in\{0,1\} \quad \forall k\in M_2.
\end{aligned}$$

(PRuvx) is solved by inspection, for a given set of multipliers $(u_a^t, v^t)$, taking $\forall i,j\in N$

$$x_{ijkm}=\begin{cases}1, & \text{for one } (i,j,k,m) \text{ s.t. } G_{ijkm}=\min\{G_{ijlp}: l,p\notin M_0,\ (i,j,l,p)\notin\theta\}\\ 0, & \text{otherwise}\end{cases} \tag{41}$$

(PRuvy) is also solved by inspection, for a given set of multipliers $u_a^t$, taking

$$y_k = 1 \text{ iff } F_a - \sum_{t \in T_y^a} u_a^t \pi_{ijkm}^{ta} \leqslant 0 \quad \forall k \in M_2. \tag{42}$$

Therefore, the optimal value of (PRuv) is

$$\begin{aligned} \mathrm{v(PRuv)} = &\sum_{a \in M_1} F_a - \sum_{t \in T_x} v^t \\ &+ \sum_i \sum_j \min_{k,m \notin M_0,\ (i,j,k,m) \notin \theta} \left\{ W_{ij} C_{ijkm} + \sum_{a \in M_2} \sum_{t \in T_y^a} u_a^t \pi_{ijkm}^{ta} + \sum_{t \in T_x} v^t \mu_{ijkm}^t \right\} \\ &+ \sum_{a \in M_2} \left( F_a - \sum_{a \in M_2} \sum_{t \in T_y^a} u_a^t \right)^-, \end{aligned}$$

where $z^- := \min\{z, 0\}$.

This value is a lower bound on the optimal value of (P), and can be used to fathom nodes of the branching tree, instead of using the lower bound given by the linear relaxation of the problem. In Figure 8, a flowchart of the solution process is shown. The details of each step of the process are specified below.

1. *Initial upper bound.* A simple greedy heuristic is used to obtain an initial upper bound. When the instance is small or difficult the heuristic solution is improved by a one-to-one interchange algorithm.
2. *Choose unresolved node.* The node with lowest lower bound among the nodes in the deepest level of the branching tree is chosen.
3. *Update upper bound.* When a terminal node is found, the associated solution is checked and, if necessary, the upper bound is updated.
4. *Iterate* (*a*). Lagrangian multipliers are updated by means of a subgradient procedure. In stage (a), only multipliers $u_a^t$ are modified. In the root node of the branching tree, the maximum number of iterations is $g \cdot 12 \cdot n$, where $g$ is 1, 2 or 3 depending on the difficulty of the instance. In the rest of the nodes, the number of iterations is $10 \cdot g$.
5. *Separate* (*a*). In stage (a), only facets (19) and (20) are separated. The reason is that these inequalities are necessary to obtain feasible solutions for the original problem. The integer optimal solution of the subproblem associated with the last iteration of the subgradient procedure, which is usually infeasible with respect to the primal problem, is considered. If, in this solution, $y_k = 0$ and either $x_{ijkm} = 1$ or $x_{ijmk} = 1$ for some $i$, $j$ and $m$, then the facet in the family (19) or (20) containing all the $x$-nodes in the same box is added to the formulation. Facets

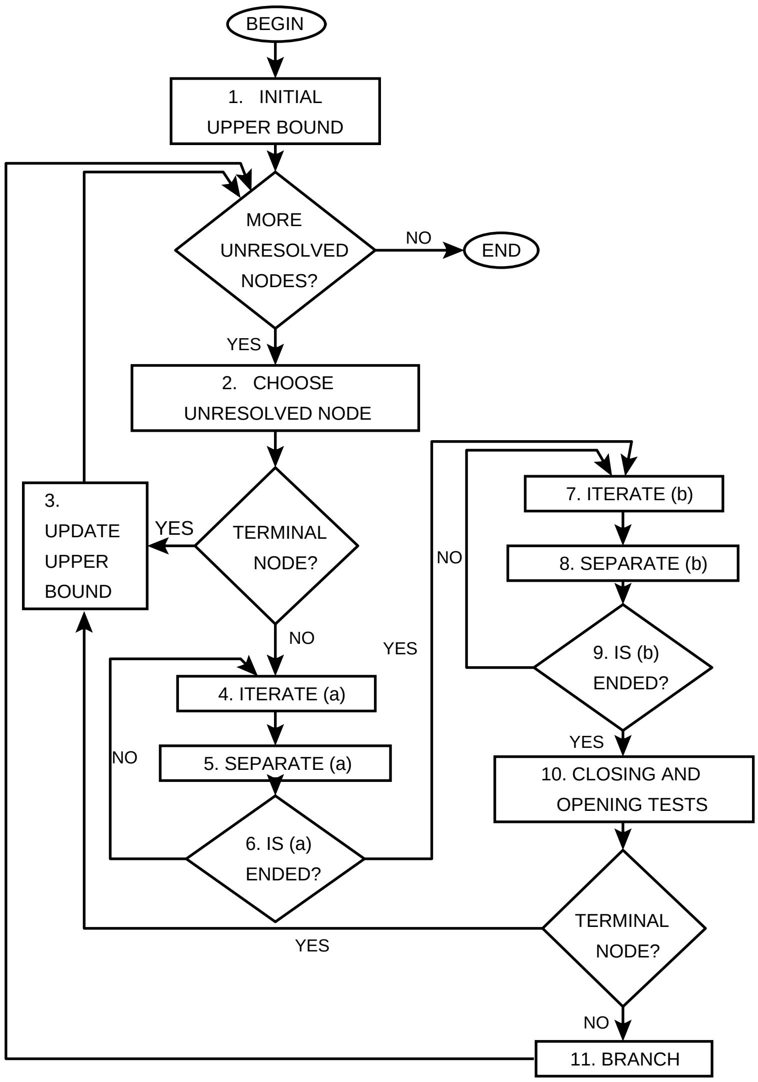


*Figure 8.* Flowchart of the solution process.

are incorporated to the formulation only in the first four levels of the branching tree, and are never removed.

6. *Is* (*a*) *ended?* If the maximum number of facets in families (19) and (20) – equal to $10000 \cdot g$ – has been reached, or no more than 50 facets have been separated in the last time the algorithm passed through step 5, end with (a).
7. *Iterate* (*b*). In stage (b), all the multipliers are updated in each iteration.
8. Separate (*b*). In (b), the facets which contain only $x$-variables are considered. In a preliminary study it was checked that some of the families

were more promising than others. Consequently, non-promising families (24 – 29), (32) and (36) are not separated by the algorithm. The separation algorithm looks for violated basic inequalities in the shape of (13), (14), (17) and (18) and randomly chooses one of the facets in families (22), (30), (31), (23), (34), (35), (33), (37) containing the variables of the basic violated inequality.

9. *Is* (*b*) *ended?* If $500 \cdot g$ facets have been separated in stage (b) or no more than 200 facets have been separated in the last time the algorithm passed through step 8, end with (b).
10. *Closing and opening tests.* Every non-fixed hub is considered. If the lower bound obtained by fixing the corresponding $y$-variable to one (resp. zero) is greater than the best upper bound, the hub is closed (resp. opened) in the node.
11. *Branch.* When the node cannot be fathomed, the variable $y_a \in M_2$ with minimum value of $F_a - \sum_{t \in T_y^a} u_a^t \pi_{ijkm}^{ta}$ is fixed to zero and one in two new nodes.

## 6. Computational Study

### 6.1. UNCAPACITATD EUCLIDEAN HUB LOCATION PROBLEM

A computational study was carried out in order to test the solution method. A set of instances commonly used in the literature, called AP (Australian Post), was downloaded from the web page *mscmga.ms.ic.ac.uk/jeb/orlib/phubinfo.html*. These instances consist of the Euclidean distances between 200 cities in Australia, a code to reduce the size of the set by grouping cities, and the values of $W_{ij}$ (postal flow between cities). For instances of size 10 to 50 there are also two kinds of fixed costs $F_k$ available. We used the more difficult sets of costs (files FcostT.n). In these instances, $N = M$. The algorithm was coded in Free Pascal under Linux, and the processor was a Mobile Pentium 4 1.7 GHz with 256 MB of RAM memory and 256 MB of swap memory.

In the tables of computational results, $L \in \{E, M, H\}$ indicates if the instance is considered Easy, Medium or Hard (this affects the value of $g$); *Hubs* is the number of hubs opened in the optimal solution; % is the percentage of difference between the initial upper bound and the optimal value; $NT$ and $T$ are the non-terminal and terminal nodes of the branching tree, respectively; $Cl$ and $Op$ are the number of hubs closed and opened by the tests across the branching tree; $Y$ is the number of facets of families (19) and (20) present in the formulation at the end of the solution process; $A$, $B$, $C$ and $D$ indicate how many facets in families (22) and (23), (30) and (34), (31) and (35), (33) and (37), respectively, have been separated. $CPU$

is the number of CPU seconds of the solution process, including data reading, preprocessing, heuristic solution, optimal solution and output.

The transportation costs $b_{ij}$, $c_{ij}$ and $d_{ij}$ were obtained by multiplying the distances between cities $D_{ij}$ by different constants. A first set of symmetric instances was built with $b_{ij} = d_{ij} = 10 \cdot D_{ij}$ and $c_{ij} = 10 \cdot \alpha \cdot D_{ij}$, where $\alpha$ is considered to be a discount factor between 0 and 1. And asymmetric instances were built with $b_{ij} = 3 \cdot D_{ij}$, $c_{ij} = 0.75 \cdot D_{ij}$ and $d_{ij} = 2 \cdot D_{ij}$. When the instance was symmetric, the number of $x$-variables was previously reduced, see Cánovas et al. (2001).

The results of the computational study for symmetric instances are shown in Table I. Instances with up to 50 points in the set $N = M$ were optimally solved. The values of $\alpha$ were taken from the set $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. Instances with great values of $\alpha$ are easier to solve because the number of $x$-variables is much reduced during the preprocessing phase. The fixed costs contained in files FcostT.$n$ led to a number of opened hubs between 3 and 9. Even though the (extremely simple) heuristic solution does not usually reach the optimal solution, the number of nodes of the branching tree was small. The reason is that the facets added to the formulation significantly improved the quality of the lower bounds, allowing the algorithm to discard many nodes either by means of the opening and closing tests, or after a few iterations of the subgradient procedure. The hardest instance, with 50 points and $\alpha = 0.1$, was solved in 61 seconds of CPU time. In Mayer and Wagner (2002), they reported many hours of CPU time for instances of 40 nodes of AP data (with unknown but equal fixed costs) in a Pentium 200 MHz. They also used another data set of the literature, the so-called CAB data. The CPU time for solving, for example, instances with 25 nodes and $\alpha = 0.6$, was around 1000 seconds; we solved, for comparison, these instances in a Pentium 133 MHz in around 6 seconds.

The results of the computational study for asymmetric instances are shown in Table II. These instances are harder than the symmetric ones. Again, the number of nodes of the branching tree was small. The hardest instance, with 50 points, was solved in 206 seconds of CPU time (although after a slight modification in the parameters of the algorithm it could be solved in 153 seconds). In Cánovas et al. (2001), Xpress was used to solve instances with up to 25 points. The advantage of using standard commercial software is obvious: it is ready to be used without additional work, and a good formulation is all that is required to solve medium sized instances. But specifically-oriented software like the algorithm developed here allows a speeding up of the process and greater instances can be solved. In fact, Cánovas et al. were able to solve instances with up to 25 points in 16 seconds in a similar machine, but greater instances exceeded the memory limitations of the software. These until now were the best results for UEHLP in the literature, but they are clearly outperformed by

*Table I.* Results for AP data, symmetric instances

| Problem | | | Solution | | Nodes | | Tests | | Facets | | | | | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | L | $\alpha$ | Hubs | % | NT | T | Cl | Op | Y | A | B | C | D | |
| | | 0.1 | 9 | 0 | 1 | 2 | 0 | 9 | 100 | 0 | 0 | 0 | 0 | 0.01 |
| | | 0.3 | 9 | 0 | 1 | 1 | 1 | 9 | 111 | 0 | 0 | 1 | 0 | 0.01 |
| 10 | E | 0.5 | 8 | 0 | 5 | 1 | 1 | 7 | 109 | 1 | 0 | 1 | 2 | 0.02 |
| | | 0.7 | 6 | 0 | 7 | 2 | 3 | 9 | 118 | 15 | 5 | 6 | 3 | 0.02 |
| | | 0.9 | 5 | 2.50 | 17 | 6 | 17 | 6 | 125 | 5 | 0 | 1 | 2 | 0.02 |
| | | 0.1 | 8 | 0.55 | 21 | 11 | 26 | 12 | 1029 | 11 | 26 | 32 | 145 | 0.93 |
| | | 0.3 | 7 | 0.79 | 23 | 6 | 34 | 16 | 1048 | 53 | 35 | 40 | 207 | 0.77 |
| 20 | E | 0.5 | 4 | 0.13 | 21 | 7 | 32 | 8 | 1075 | 84 | 25 | 23 | 129 | 0.46 |
| | | 0.7 | 4 | 0.43 | 15 | 1 | 18 | 2 | 1044 | 60 | 2 | 2 | 36 | 0.21 |
| | | 0.9 | 4 | 0 | 5 | 2 | 14 | 3 | 924 | 16 | 0 | 0 | 0 | 0.10 |
| | | 0.1 | 6 | 0.04 | 13 | 3 | 24 | 10 | 3602 | 67 | 63 | 63 | 408 | 4.45 |
| | | 0.3 | 4 | 0.09 | 9 | 6 | 31 | 7 | 3737 | 90 | 49 | 40 | 322 | 3.24 |
| 30 | E | 0.5 | 4 | 1.87 | 23 | 10 | 49 | 6 | 3405 | 108 | 31 | 15 | 377 | 3.26 |
| | | 0.7 | 4 | 1.09 | 9 | 2 | 27 | 1 | 3309 | 247 | 8 | 11 | 106 | 1.43 |
| | | 0.9 | 4 | 0.63 | 7 | 2 | 33 | 3 | 3294 | 116 | 0 | 2 | 18 | 1.10 |
| | | 0.1 | 5 | 0 | 33 | 5 | 74 | 22 | 10155 | 77 | 30 | 40 | 756 | 27.25 |
| | | 0.3 | 4 | 0 | 11 | 7 | 43 | 4 | 10117 | 100 | 21 | 21 | 426 | 13.04 |
| 40 | E | 0.5 | 3 | 0 | 13 | 4 | 46 | 7 | 9897 | 191 | 13 | 23 | 357 | 10.83 |
| | | 0.7 | 3 | 0 | 3 | 4 | 40 | 3 | 9374 | 365 | 9 | 8 | 193 | 6.32 |
| | | 0.9 | 3 | 0 | 7 | 2 | 36 | 2 | 8404 | 309 | 0 | 0 | 28 | 4.15 |
| | | 0.1 | 4 | 4.53 | 7 | 2 | 45 | 1 | 20311 | 204 | 70 | 63 | 1110 | 61.37 |
| | | 0.3 | 4 | 2.57 | 5 | 2 | 45 | 2 | 20155 | 266 | 31 | 25 | 814 | 45.95 |
| 50 | M | 0.5 | 4 | 1.42 | 7 | 1 | 46 | 1 | 20020 | 544 | 19 | 15 | 646 | 38.61 |
| | | 0.7 | 4 | 0.01 | 7 | 2 | 47 | 3 | 19972 | 818 | 9 | 8 | 185 | 37.66 |
| | | 0.9 | 3 | 0 | 9 | 3 | 55 | 8 | 18454 | 367 | 2 | 5 | 12 | 20.84 |

*Table II.* Results for AP data, asymmetric instances

| Problem | | Solution | | Nodes | | Tests | | Facets | | | | | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | L | Hubs | % | NT | T | Cl | Op | Y | A | B | C | D | |
| 10 | E | 3 | 0 | 3 | 1 | 6 | 3 | 438 | 13 | 2 | 2 | 6 | 0.02 |
| 20 | E | 2 | 0 | 3 | 2 | 17 | 1 | 3704 | 107 | 0 | 1 | 42 | 1.06 |
| 30 | E | 2 | 0 | 3 | 1 | 27 | 2 | 10532 | 479 | 19 | 15 | 277 | 4.85 |
| 40 | M | 1 | 0 | 37 | 12 | 74 | 38 | 20414 | 432 | 11 | 15 | 802 | 81.84 |
| 50 | H | 2 | 1.49 | 63 | 28 | 506 | 2 | 31735 | 547 | 112 | 97 | 1155 | 205.61 |

our new algorithm. Very recently, in Boland et al. (2003), they reported more than 13 hours of CPU time for the instance of Table II with 50 nodes, solving it in a Digital Personal workstation with a 500 MHz alpha chip.

Since the number of opened hubs is very small, we also considered the same instances but dividing the fixed costs $F_k$ by 10. The results are shown in Table III.

*Table III.* Results for modified AP data, asymmetric instances

| Problem | | Solution | | Nodes | | Tests | | Facets | | | | | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | L | Hubs | % | NT | T | Cl | Op | Y | A | B | C | D | |
| 10 | E | 10 | 0 | 1 | 1 | 0 | 10 | 171 | 0 | 0 | 0 | 0 | 0.01 |
| 20 | E | 14 | 0 | 5 | 4 | 3 | 15 | 1139 | 1 | 7 | 5 | 10 | 0.59 |
| 30 | E | 10 | 0.42 | 9 | 1 | 20 | 6 | 4037 | 81 | 125 | 121 | 315 | 4.42 |
| 40 | M | 8 | 1.29 | 37 | 12 | 74 | 38 | 12760 | 96 | 68 | 78 | 813 | 55.14 |
| 50 | M | 7 | 3.19 | 21 | 6 | 201 | 25 | 21977 | 264 | 73 | 80 | 1009 | 126.51 |

### 6.2. $p$-HUB MEDIAN PROBLEM

With the aim of widening the comparison, the same instances but with the additional constraint of forcing the number of opened hubs, and fixed costs $F_k = 0\ \forall k$, were considered. This *p-median version* of the problem is usually known as $p$-hub median problem. Note that the polyhedron associated with this problem is not the same, and hence the valid inequalities used to enforce the formulation are not facets. In any case, although further analysis can give better constraints for the $p$-hub median problem, our inequalities remain valid and the computational test can be carried out after making the corresponding modifications in the calculus of the lower bound and the management of the branching tree.

Tables IV and V show the results for symmetric instances of the $p$-hub median problem. The same instances with up to 50 points and values of $p$ between 3 and 8 and $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ were solved. The opening and closing tests did not work well for these instances and were suppressed. The number of nodes of the branching tree was higher here, and it was necessary to classify the instances with 40 points as *medium difficult*, i.e., to use more iterations in the subgradient method and a greater maximum number of facets. But, all in all, the instances were optimally solved and the algorithm did not require a high computational effort.

In Table VI, the results for asymmetric instances of the $p$-hub median problem are shown. It is evident that, for instances with small values of $p$ and consequently with few feasible solutions, the procedure works relatively worse than for instances with greater values of $p$. The reason is that, for small values of $p$, the valid inequalities developed for UEHLP need to be strengthened by considering the $p$-median constraint. This is a matter for future research, and there exists a previous work on the polytopes of several $p$-median versions of several location problems that could be adapted to work with our strengthened formulation of the $p$-median hub location problem, see e.g. Avella and Sassano (2001).

On the other hand, for high values of $p$, we have not found any procedure in the literature that solve the problem more efficiently. The results

*Table IV.* Results for AP data, symmetric instances, fixed number of hubs (1st part)

| Problem | | | Solution | | Nodes | | Facets | | | | | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *n* | $\alpha$ | L | *p* | % | NT | T | Y | A | B | C | D | |
| | | | 3 | 0.21 | 15 | 4 | 157 | 9 | 2 | 4 | 16 | 0.02 |
| | | | 4 | 0.54 | 23 | 6 | 151 | 9 | 6 | 11 | 18 | 0.05 |
| 10 | 0.1 | E | 5 | 2.31 | 19 | 2 | 98 | 12 | 1 | 1 | 7 | 0.03 |
| | | | 6 | 0 | 62 | 7 | 40 | 0 | 3 | 6 | 1 | 0.02 |
| | | | 7 | 0 | 67 | 14 | 30 | 0 | 0 | 0 | 0 | 0.02 |
| | | | 8 | 0 | 36 | 25 | 20 | 0 | 0 | 0 | 0 | 0.02 |
| | | | 3 | 0 | 21 | 4 | 148 | 6 | 0 | 0 | 8 | 0.02 |
| | | | 4 | 1.11 | 33 | 8 | 105 | 23 | 3 | 3 | 11 | 0.04 |
| 10 | 0.3 | E | 5 | 1.27 | 18 | 5 | 90 | 10 | 1 | 0 | 3 | 0.02 |
| | | | 6 | 0 | 59 | 6 | 40 | 1 | 7 | 10 | 0 | 0.01 |
| | | | 7 | 0 | 66 | 13 | 30 | 0 | 0 | 0 | 0 | 0.00 |
| | | | 8 | 0 | 36 | 23 | 20 | 0 | 0 | 0 | 0 | 0.00 |
| | | | 3 | 0 | 34 | 9 | 114 | 21 | 2 | 4 | 5 | 0.02 |
| | | | 4 | 0 | 22 | 3 | 108 | 17 | 0 | 0 | 7 | 0.03 |
| 10 | 0.5 | E | 5 | 0 | 13 | 2 | 90 | 5 | 3 | 6 | 3 | 0.02 |
| | | | 6 | 0 | 59 | 6 | 40 | 1 | 2 | 3 | 0 | 0.01 |
| | | | 7 | 0 | 66 | 15 | 30 | 0 | 0 | 0 | 0 | 0.01 |
| | | | 8 | 0 | 36 | 23 | 20 | 0 | 0 | 0 | 0 | 0.01 |
| | | | 3 | 0 | 30 | 5 | 112 | 13 | 2 | 4 | 1 | 0.02 |
| | | | 4 | 0.39 | 14 | 7 | 120 | 10 | 0 | 2 | 0 | 0.00 |
| 10 | 0.7 | E | 5 | 0.44 | 11 | 2 | 113 | 3 | 0 | 0 | 2 | 0.01 |
| | | | 6 | 0 | 56 | 1 | 40 | 2 | 0 | 1 | 0 | 0.02 |
| | | | 7 | 0 | 66 | 13 | 171 | 30 | 0 | 0 | 0 | 0.02 |
| | | | 8 | 0 | 36 | 23 | 20 | 0 | 0 | 0 | 0 | 0.01 |
| | | | 3 | 0 | 26 | 1 | 114 | 13 | 0 | 1 | 0 | 0.02 |
| | | | 4 | 0.22 | 19 | 4 | 121 | 4 | 0 | 0 | 1 | 0.02 |
| 10 | 0.9 | E | 5 | 0.43 | 11 | 2 | 95 | 3 | 0 | 0 | 0 | 0.00 |
| | | | 6 | 0 | 56 | 1 | 40 | 1 | 0 | 2 | 0 | 0.01 |
| | | | 7 | 0 | 57 | 16 | 30 | 0 | 0 | 0 | 0 | 0.01 |
| | | | 8 | 0 | 30 | 19 | 20 | 0 | 0 | 0 | 0 | 0.00 |
| | | | 3 | 0 | 34 | 1 | 1352 | 34 | 27 | 12 | 125 | 1.12 |
| | | | 4 | 0 | 34 | 1 | 1171 | 79 | 17 | 13 | 140 | 1.02 |
| 20 | 0.1 | E | 5 | 0 | 51 | 2 | 894 | 37 | 24 | 17 | 144 | 1.14 |
| | | | 6 | 0 | 32 | 1 | 786 | 15 | 23 | 24 | 86 | 0.71 |
| | | | 7 | 0 | 27 | 2 | 765 | 4 | 21 | 22 | 47 | 0.66 |
| | | | 8 | 0.75 | 50 | 3 | 730 | 4 | 8 | 6 | 43 | 0.86 |
| | | | 3 | 0 | 38 | 1 | 1146 | 58 | 14 | 18 | 165 | 0.73 |
| | | | 4 | 0 | 42 | 1 | 1053 | 46 | 23 | 18 | 119 | 0.75 |
| 20 | 0.3 | E | 5 | 0 | 55 | 6 | 986 | 33 | 7 | 12 | 65 | 0.75 |
| | | | 6 | 0.31 | 40 | 3 | 788 | 39 | 23 | 15 | 66 | 0.56 |
| | | | 7 | 0 | 48 | 3 | 701 | 17 | 11 | 6 | 52 | 0.57 |
| | | | 8 | 0 | 72 | 5 | 666 | 13 | 14 | 8 | 55 | 0.76 |
| | | | 3 | 0 | 41 | 10 | 1109 | 30 | 5 | 4 | 98 | 0.57 |
| | | | 4 | 0 | 72 | 1 | 1007 | 56 | 9 | 7 | 87 | 0.65 |
| 20 | 0.5 | E | 5 | 0.06 | 129 | 4 | 913 | 30 | 9 | 3 | 81 | 0.87 |
| | | | 6 | 0 | 50 | 3 | 751 | 14 | 14 | 13 | 59 | 0.41 |
| | | | 7 | 0 | 32 | 1 | 694 | 21 | 10 | 13 | 68 | 0.33 |
| | | | 8 | 0 | 72 | 5 | 591 | 24 | 10 | 12 | 38 | 0.49 |
| | | | 3 | 0 | 55 | 16 | 965 | 56 | 4 | 3 | 41 | 0.38 |
| | | | 4 | 0 | 69 | 2 | 891 | 70 | 8 | 8 | 50 | 0.47 |

*Table IV.* (Continued)

| Problem | | | | Solution | Nodes | | Facets | | | | | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | L | $p$ | % | NT | T | Y | A | B | C | D | |
| 20 | 0.7 | E | 5 | 0.27 | 184 | 11 | 765 | 62 | 11 | 12 | 72 | 0.82 |
| | | | 6 | 0.88 | 140 | 7 | 691 | 89 | 12 | 13 | 67 | 0.77 |
| | | | 7 | 0.60 | 150 | 9 | 696 | 53 | 17 | 15 | 58 | 0.73 |
| | | | 8 | 0.35 | 150 | 5 | 628 | 56 | 16 | 11 | 46 | 0.70 |
| | | | 3 | 0 | 87 | 22 | 806 | 29 | 0 | 3 | 5 | 0.27 |
| | | | 4 | 0 | 217 | 14 | 674 | 53 | 3 | 2 | 7 | 0.63 |
| 20 | 0.9 | E | 5 | 0.24 | 580 | 13 | 628 | 55 | 12 | 13 | 11 | 1.56 |
| | | | 6 | 0.97 | 776 | 19 | 594 | 106 | 20 | 15 | 25 | 2.26 |
| | | | 7 | 0.24 | 258 | 9 | 561 | 99 | 12 | 10 | 18 | 0.79 |
| | | | 8 | 0.26 | 268 | 15 | 570 | 108 | 17 | 20 | 35 | 1.06 |
| | | | 3 | 7.61 | 355 | 66 | 4699 | 118 | 43 | 50 | 521 | 49.99 |
| | | | 4 | 7.25 | 828 | 47 | 3915 | 88 | 92 | 98 | 336 | 76.74 |
| 30 | 0.1 | E | 5 | 8.12 | 558 | 17 | 3428 | 50 | 27 | 27 | 547 | 56.39 |
| | | | 6 | 5.36 | 565 | 8 | 2814 | 57 | 72 | 63 | 335 | 45.31 |
| | | | 7 | 2.38 | 288 | 5 | 2459 | 76 | 102 | 93 | 330 | 23.91 |
| | | | 8 | 2.01 | 221 | 6 | 2363 | 52 | 63 | 57 | 394 | 18.73 |
| | | | 3 | 5.55 | 483 | 54 | 3737 | 104 | 26 | 23 | 385 | 37.76 |
| | | | 4 | 5.74 | 573 | 20 | 3609 | 143 | 41 | 56 | 448 | 42.38 |
| 30 | 0.3 | E | 5 | 5.36 | 804 | 21 | 2843 | 111 | 62 | 61 | 365 | 48.22 |
| | | | 6 | 2.92 | 385 | 6 | 2538 | 107 | 43 | 28 | 344 | 23.24 |
| | | | 7 | 1.10 | 129 | 4 | 2328 | 66 | 93 | 90 | 319 | 8.32 |
| | | | 8 | 1.18 | 142 | 7 | 2319 | 51 | 80 | 74 | 284 | 9.30 |
| | | | 3 | 3.32 | 176 | 31 | 3782 | 233 | 9 | 8 | 302 | 11.50 |
| | | | 4 | 4.17 | 480 | 17 | 3224 | 170 | 29 | 25 | 366 | 24.89 |
| 30 | 0.5 | E | 5 | 3.60 | 851 | 14 | 2740 | 138 | 58 | 51 | 263 | 35.60 |
| | | | 6 | 1.41 | 199 | 6 | 2538 | 116 | 41 | 36 | 238 | 9.67 |
| | | | 7 | 0.69 | 163 | 8 | 2375 | 46 | 44 | 31 | 147 | 7.16 |
| | | | 8 | 0.79 | 214 | 11 | 1945 | 22 | 36 | 33 | 132 | 8.02 |
| | | | 3 | 1.66 | 172 | 35 | 3540 | 214 | 11 | 4 | 140 | 7.02 |
| | | | 4 | 2.76 | 605 | 24 | 2968 | 243 | 31 | 37 | 325 | 21.48 |
| 30 | 0.7 | E | 5 | 2.73 | 877 | 12 | 2592 | 253 | 48 | 43 | 235 | 28.86 |
| | | | 6 | 0.73 | 400 | 7 | 2314 | 153 | 47 | 31 | 212 | 12.18 |
| | | | 7 | 0.72 | 518 | 13 | 1999 | 184 | 74 | 58 | 228 | 14.24 |
| | | | 8 | 0.79 | 341 | 14 | 2053 | 169 | 70 | 64 | 236 | 11.45 |
| | | | 3 | 1.43 | 153 | 26 | 3156 | 77 | 0 | 0 | 11 | 4.12 |
| | | | 4 | 2.25 | 563 | 68 | 2799 | 170 | 0 | 0 | 56 | 12.69 |
| 30 | 0.9 | E | 5 | 2.25 | 2272 | 105 | 2163 | 330 | 20 | 17 | 79 | 43.70 |
| | | | 6 | 0.60 | 427 | 08 | 2097 | 271 | 34 | 23 | 112 | 9.53 |
| | | | 7 | 0.53 | 1112 | 23 | 1923 | 299 | 32 | 26 | 127 | 22.80 |
| | | | 8 | 0.62 | 1241 | 4 | 1868 | 270 | 57 | 43 | 132 | 25.09 |

in Ernst and Krishnamoorthy (1998) are very good when the number of hubs opened is small, but the time needed to solve the instances grows exponentially with $p$. The reported times for the AP instances with $n = 50$ and $p = 3, 4, 5$ are 49, 694 and 7545 seconds (in a 200 MHz machine), and for $p = 10$ the time increased to 57243 seconds (about 16 hours). We solved this instance, for comparison, in a Pentium 133 with 78 MB of memory in

*Table V.* Results for AP data, symmetric instances, fixed number of hubs (2nd part)

| Problem | | | | Solution | Nodes | | Facets | | | | | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | L | $p$ | % | NT | T | Y | A | B | C | D | |
| | | | 3 | 0 | 74 | 1 | 11177 | 235 | 145 | 132 | 547 | 81.68 |
| | | | 4 | 0.14 | 132 | 3 | 9462 | 96 | 56 | 57 | 830 | 100.34 |
| 40 | 0.1 | M | 5 | 0 | 82 | 1 | 8368 | 70 | 56 | 48 | 938 | 80.69 |
| | | | 6 | 0 | 75 | 2 | 7355 | 99 | 120 | 124 | 789 | 68.56 |
| | | | 7 | 0 | 104 | 1 | 6914 | 79 | 135 | 144 | 795 | 82.92 |
| | | | 8 | 0 | 79 | 2 | 6126 | 56 | 121 | 210 | 686 | 67.43 |
| | | | 3 | 0 | 74 | 1 | 10234 | 373 | 58 | 57 | 780 | 61.97 |
| | | | 4 | 0 | 73 | 2 | 8228 | 96 | 14 | 12 | 888 | 49.05 |
| 40 | 0.3 | M | 5 | 0 | 84 | 1 | 7355 | 156 | 46 | 42 | 838 | 51.45 |
| | | | 6 | 0 | 70 | 1 | 6835 | 141 | 89 | 97 | 901 | 47.80 |
| | | | 7 | 0.07 | 98 | 5 | 6341 | 129 | 57 | 60 | 778 | 55.26 |
| | | | 8 | 0.43 | 148 | 3 | 5318 | 124 | 82 | 80 | 746 | 67.43 |
| | | | 3 | 0 | 75 | 2 | 9185 | 678 | 20 | 20 | 583 | 42.50 |
| | | | 4 | 0 | 73 | 2 | 7723 | 160 | 29 | 14 | 820 | 37.71 |
| 40 | 0.5 | M | 5 | 0 | 70 | 1 | 6858 | 240 | 64 | 87 | 638 | 33.30 |
| | | | 6 | 0 | 68 | 1 | 6091 | 225 | 60 | 42 | 706 | 31.88 |
| | | | 7 | 0.19 | 73 | 2 | 5427 | 221 | 49 | 59 | 824 | 34.27 |
| | | | 8 | 0.39 | 109 | 4 | 5517 | 100 | 60 | 59 | 546 | 41.75 |
| | | | 3 | 0 | 110 | 1 | 8236 | 729 | 3 | 1 | 406 | 33.02 |
| | | | 4 | 0 | 82 | 1 | 6773 | 379 | 27 | 22 | 616 | 24.43 |
| 40 | 0.7 | M | 5 | 0.38 | 119 | 2 | 6444 | 232 | 42 | 43 | 498 | 32.62 |
| | | | 6 | 0.31 | 169 | 4 | 5643 | 383 | 52 | 50 | 584 | 38.93 |
| | | | 7 | 0 | 110 | 1 | 4900 | 154 | 17 | 17 | 344 | 25.78 |
| | | | 8 | 0.29 | 190 | 3 | 4661 | 204 | 44 | 31 | 469 | 37.64 |
| | | | 3 | 0 | 106 | 1 | 7316 | 180 | 2 | 2 | 19 | 17.44 |
| | | | 4 | 0 | 102 | 1 | 5814 | 200 | 1 | 0 | 51 | 14.89 |
| 40 | 0.9 | M | 5 | 0.24 | 158 | 3 | 5548 | 160 | 6 | 6 | 80 | 17.62 |
| | | | 6 | 0 | 150 | 1 | 4789 | 250 | 21 | 10 | 110 | 20.66 |
| | | | 7 | 0 | 266 | 1 | 4406 | 257 | 15 | 7 | 129 | 33.11 |
| | | | 8 | 0.20 | 480 | 9 | 4132 | 194 | 17 | 12 | 141 | 45.76 |
| | | | 3 | 0 | 103 | 2 | 20039 | 160 | 90 | 77 | 1011 | 186.38 |
| | | | 4 | 0.37 | 135 | 6 | 19821 | 124 | 105 | 100 | 1397 | 313.67 |
| 50 | 0.1 | M | 5 | 0 | 113 | 2 | 15269 | 65 | 77 | 83 | 1109 | 211.22 |
| | | | 6 | 0 | 114 | 1 | 13730 | 88 | 142 | 146 | 857 | 185.45 |
| | | | 7 | 0 | 138 | 7 | 13001 | 87 | 155 | 165 | 643 | 199.10 |
| | | | 8 | 0 | 149 | 2 | 12269 | 49 | 170 | 164 | 721 | 226.08 |
| | | | 3 | 0 | 97 | 6 | 20085 | 231 | 68 | 69 | 898 | 159.35 |
| | | | 4 | 0 | 114 | 3 | 17207 | 73 | 116 | 101 | 833 | 159.00 |
| 50 | 0.3 | M | 5 | 0.01 | 134 | 3 | 14415 | 136 | 51 | 51 | 932 | 152.33 |
| | | | 6 | 0 | 96 | 1 | 13302 | 119 | 33 | 37 | 978 | 127.60 |
| | | | 7 | 0 | 97 | 2 | 11943 | 124 | 76 | 82 | 805 | 138.77 |
| | | | 8 | 0 | 86 | 1 | 11054 | 82 | 127 | 114 | 773 | 130.86 |
| | | | 3 | 0 | 99 | 2 | 18846 | 503 | 3 | 4 | 505 | 129.66 |
| | | | 4 | 0 | 98 | 1 | 15535 | 170 | 34 | 39 | 522 | 783.11 |
| 50 | 0.5 | M | 5 | 0 | 136 | 1 | 12865 | 206 | 41 | 36 | 928 | 107.66 |
| | | | 6 | 0 | 146 | 1 | 12196 | 161 | 30 | 23 | 883 | 110.99 |
| | | | 7 | 0.18 | 415 | 4 | 10853 | 167 | 63 | 45 | 732 | 232.74 |
| | | | 8 | 0.24 | 413 | 4 | 10028 | 206 | 43 | 41 | 730 | 230.42 |
| | | | 3 | 0 | 108 | 1 | 16624 | 727 | 5 | 2 | 290 | 85.44 |
| | | | 4 | 0 | 116 | 1 | 13459 | 438 | 87 | 96 | 644 | 73.92 |

*Table V.* (Continued)

| Problem | | | | Solution | Nodes | | Facets | | | | | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | L | $p$ | % | NT | T | Y | A | B | C | D | |
| 50 | 0.7 | M | 5 | 0 | 170 | 1 | 12121 | 256 | 25 | 20 | 722 | 86.41 |
| | | | 6 | 0.21 | 273 | 4 | 11080 | 257 | 45 | 49 | 653 | 121.47 |
| | | | 7 | 0 | 503 | 4 | 9949 | 308 | 34 | 30 | 701 | 194.07 |
| | | | 8 | 0.03 | 433 | 4 | 9210 | 367 | 55 | 51 | 788 | 177.59 |
| | | | 3 | 0.96 | 331 | 78 | 14036 | 293 | 0 | 0 | 58 | 88.83 |
| | | | 4 | 0 | 214 | 7 | 11213 | 418 | 22 | 31 | 117 | 35.79 |
| 50 | 0.9 | M | 5 | 0 | 246 | 3 | 10452 | 293 | 6 | 6 | 181 | 61.18 |
| | | | 6 | 0 | 308 | 1 | 9602 | 361 | 20 | 17 | 253 | 79.30 |
| | | | 7 | 0 | 927 | 2 | 8745 | 409 | 27 | 9 | 277 | 196.93 |
| | | | 8 | 0.16 | 807 | 2 | 8176 | 409 | 40 | 26 | 274 | 187.98 |

*Table VI.* Results for AP data, asymmetric instances, fixed number of hubs

| Problem | | | Solution | Nodes | | Facets | | | | | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | L | $p$ | % | NT | T | Y | A | B | C | D | |
| | | 3 | 0 | 15 | 4 | 264 | 38 | 5 | 3 | 23 | 0.03 |
| | | 4 | 0 | 16 | 1 | 234 | 15 | 7 | 4 | 20 | 0.04 |
| 10 | E | 5 | 0 | 10 | 1 | 215 | 15 | 0 | 3 | 5 | 0.03 |
| | | 6 | 0 | 14 | 7 | 198 | 0 | 5 | 6 | 5 | 0.01 |
| | | 7 | 0 | 6 | 1 | 189 | 5 | 0 | 0 | 1 | 0.01 |
| | | 8 | 0 | 36 | 23 | 38 | 0 | 0 | 0 | 0 | 0.02 |
| | | 3 | 0 | 54 | 7 | 2277 | 176 | 25 | 20 | 340 | 2.45 |
| | | 4 | 0 | 45 | 2 | 2029 | 159 | 50 | 40 | 330 | 2.13 |
| 20 | E | 5 | 0 | 46 | 3 | 1895 | 88 | 19 | 18 | 225 | 1.96 |
| | | 6 | 0.38 | 41 | 2 | 1716 | 56 | 31 | 31 | 162 | 1.76 |
| | | 7 | 0.01 | 46 | 3 | 1306 | 23 | 21 | 25 | 92 | 1.41 |
| | | 8 | 0.03 | 46 | 3 | 1265 | 14 | 21 | 13 | 124 | 1.50 |
| | | 3 | 0 | 54 | 1 | 7950 | 536 | 72 | 70 | 521 | 28.30 |
| | | 4 | 0 | 52 | 1 | 7162 | 241 | 214 | 208 | 621 | 28.32 |
| 30 | M | 5 | 0 | 53 | 2 | 4938 | 228 | 41 | 39 | 694 | 20.53 |
| | | 6 | 0 | 52 | 7 | 5003 | 77 | 84 | 72 | 329 | 17.36 |
| | | 7 | 0 | 46 | 1 | 4760 | 125 | 94 | 94 | 504 | 17.56 |
| | | 8 | 0 | 56 | 3 | 3784 | 62 | 50 | 43 | 251 | 15.24 |
| | | 3 | 0 | 74 | 1 | 20030 | 445 | 54 | 38 | 886 | 114.76 |
| | | 4 | 0 | 74 | 1 | 17003 | 142 | 73 | 73 | 1248 | 116.84 |
| 40 | M | 5 | 0 | 96 | 1 | 15236 | 86 | 19 | 27 | 877 | 118.89 |
| | | 6 | 0 | 71 | 2 | 13280 | 229 | 51 | 39 | 732 | 91.42 |
| | | 7 | 0.07 | 97 | 8 | 12114 | 161 | 129 | 119 | 698 | 97.33 |
| | | 8 | 0 | 75 | 2 | 11795 | 119 | 79 | 68 | 804 | 97.79 |
| | | 3 | 0 | 304 | 1 | 30924 | 590 | 104 | 104 | 2177 | 1004.76 |
| | | 4 | 0 | 148 | 1 | 30068 | 227 | 104 | 88 | 1417 | 567.96 |
| 50 | H | 5 | 0 | 99 | 4 | 27676 | 239 | 94 | 86 | 1093 | 386.24 |
| | | 6 | 0 | 89 | 2 | 25889 | 175 | 93 | 71 | 1181 | 352.35 |
| | | 7 | 0.04 | 105 | 2 | 23776 | 220 | 108 | 114 | 1243 | 349.02 |
| | | 8 | 0 | 84 | 1 | 21831 | 250 | 147 | 129 | 1624 | 311.55 |

2775 seconds (about 46 minutes). Boland et al. (2003) needed between 2.5 and 19 hours of CPU time for the instances with 50 points and $p$ between 2 and 5.

## 7. Conclusions

Formulating a real model as an integer programming problem is a task that can be performed in many ways. It is mandatory to look for a set of adequate variables and to strengthen the constraints as much as possible, in order to obtain reduced and useful formulations. The evolution in the literature on uncapacitated hub location problems can be shown as a clear example of how the improvement of the formulations has led to an impressive reduction in the computational times and the sizes of the approachable instances. Another step in this direction has been given in this paper, by discarding a part of the set of feasible solutions since the optimal solution was proved to be outside the set.

By integrating the analysis of the polyhedron associated with this strengthened formulation and the well-known Lagrangian relaxation technique, a very efficient relax-and-cut algorithm for the Uncapacitated Euclidean Multiple Allocation Hub Location Problem has been implemented. This problem arises in transportation systems when several locations send and receive passengers and/or express packages and the performance of these systems can be improved by using hubs, where the passengers/packages are collected and distributed. Greater instances, where the transportation costs between hubs satisfy the triangle inequality, have been optimally solved in minutes, even for the $p$-median version of the problem.

As a matter of future research, good heuristic algorithms are required to approximate the optimal solution of instances with hundreds of nodes. Capacitated and more general versions of the problem are also being studied, but there is still a lot of work to be done in this line. The main ideas developed in this paper could be used to approach other versions of the problem and other related location/transportation problems.

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